トロピカル射影平面曲線の交点の有理性

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Abstract

Tropical geometry is a branch of algebraic geometry considered over the tropical semiring with max-plus algebra, bridging diverse mathematical fields such as combinatorics and applied mathematics. In this study, we focus on the stable intersection of finite parts of two tropical projective plane curves determined by tropical homogeneous polynomials in three variables. Our findings reveal the Γ_{val}-rationality of these stable intersection points.

1 Introduction

For a "nonzero" specific tropical polynomial f with at least two terms, duality [1] depicts the corelations between its tropical hypersurface and regular subdivision induced by coefficients on the lattice points of Newton polytope $\mathrm{Newt}(f)$. Taking this property, we can efficiently draw tropical plane curves.

Kapranov et al. [3] established a connection between tropical hypersurfaces and the tropicalization of classical algebraic hypersurfaces over an algebraically closed field equipped with a nontrivial valuation.

Over a possibly not algebraically closed field with a nontrivial valuation, the correspondence in Kapranov ' s theorem remains unclear. In this study, we focus on the stable intersection of finite parts of tropical projective plane curves, revealing its composition and the rationality of its points. Based on these findings, we strive to provide a foundation for further discussions.

2 Preliminaries

We introduce some fundamental concepts and basic facts in tropical geometry that will be used in the rest of the text. For more details and proofs, we refer to expositions in [1], [2], [5], [7], [10].

2.1 Tropical algebra and hypersurface

Definition 2.1. $(\mathbf{T} := \mathbf{R} \cup \{-\infty\}, \oplus, \odot)$ is called the *tropical semifield* with two tropical operations as:

$$x \oplus y := \max(x, y), x \odot y := x + y, \text{ where } x, y \in \mathbf{T}.$$

$\oplus$ and $\odot$ are called *tropical addition* and *tropical multiplication* respectively.

Note that in order to distinguish from common plus and times, frequently they can be written as "$x + y$" and "xy" respectively. The neutral element for tropical addition (resp. tropical multiplication) is $-\infty$ (resp. 0). There do not exist addictive inverses.

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Definition 2.2. The finite formal sum

$$a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0$$

where $n \in \mathbf{N}$ and coefficient $a_i \in \mathbf{T}$ is called a *tropical polynomial* in x with $a_i \in \mathbf{T}$. The term $a_i \odot x^i$ is called *tropical term* in x. If $a_n \neq -\infty$, then the polynomial is said to be of degree n, denoted as $\deg(f) = n$.

Definition 2.3. The set of all polynomials above constitutes a semiring, called the *tropical polynomial semiring* in a variable x with coefficients in $\mathbf{T}$, denoted by $\mathbf{T}[x]$. The tropical polynmial semiring $\mathbf{T}[x_1, \ldots, x_n]$ in variables $x_1, \ldots, x_n$ with coefficients in $\mathbf{T}$ is induced by

$$\mathbf{T}[x_1, \ldots, x_n] := (\mathbf{T}[x_1, \ldots, x_{n-1}])[x_n].$$

The definitions of term and degree of multivariate version are analogues to definition 2.2.

Remark 2.4. We can also define some similar concepts for *tropical Laurent polynomials.* For simplicity, we restrict our consideration to tropical polynomials.

Remark 2.5. Formal tropical polynomials are not always identified with functionally equivalent polynomials. For instance, let $f(x) =$“$x^2 + 1x + 2$”, $g(x) =$“$x^2 + 0x + 2$”. $f(x) \neq g(x)$ but for every $p \in \mathbf{T}$, we have $f(p) = g(p)$.

Definition 2.6. The *tropical polynomial function semiring* in n variables over $\mathbf{T}$ is the quotient semiring $PF(\mathbf{T}^n) := \mathbf{T}[x_1, \ldots, x_n]/\sim$, where

$$f \sim g \iff f(p) = g(p), \forall p \in \mathbf{T}^n$$

An element in $PF(\mathbf{T}^n)$ is called a *tropical polynomial function.*

Remark 2.7. In this study, a tropical polynomial refers to a “nonzero” tropical polynomial function with at least two terms.

Definition 2.8. For a given tropical polynomial $f =$“$\sum_{\mathbf{u} \in \mathbf{N}^n} c_{\mathbf{u}} x^{\mathbf{u}}$” in $\mathbf{T}[x_1, \ldots, x_n]$, the *tropical hypersurface* $V_{\mathbf{R}^n}(f)$ determined by f is defined as the set of points where at least two distinct terms of f achieve the maximum, i.e.,

$$V_{\mathbf{R}^n}(f) := \{p \in \mathbf{R}^n : \exists \mathbf{i} \neq \mathbf{j} \in \mathbf{N}^n, f(p) = c_{\mathbf{i}} p^{\mathbf{i}} = c_{\mathbf{j}} p^{\mathbf{j}}\}.$$

When $n = 2$, $V_{\mathbf{R}^2}(f)$ is called *tropical plane curve.*

Definition 2.9. *Tropical projective plane* $\mathbf{TP}^2$ is a quotient space induced by equivalence relation where corresponding coordinates differ by a constant number in $\mathbf{T}^\times$. It can be glued from three affine patches $U_i := \{(x_1 : x_2 : x_3) \in \mathbf{TP}^2 : x_i \neq -\infty\} \cong \mathbf{T}^2$, $i = 1, 2, 3$.

Definition 2.10. For a given homogeneous tropical polynomial $F =$“$\sum_{\mathbf{u} \in \mathbf{N}^n} c_{\mathbf{u}} x^{\mathbf{u}}$” in $\mathbf{T}[x_1, x_2, x_3]$, the *tropical projective plane curve* $V_{\mathbf{TP}^2}(F)$ determined by F is defined as

$$V_{\mathbf{TP}^2}(F) := \{p \in \mathbf{TP}^2 : \exists \mathbf{i} \neq \mathbf{j} \in \mathbf{N}^n, F(p) = c_{\mathbf{i}} p^{\mathbf{i}} = c_{\mathbf{j}} p^{\mathbf{j}}\}.$$

It is not difficult to check this is well-defined. In the book [2] by Mikhalkin and Rau, it is shown that $V_{\mathbf{TP}^2}(F)$ consists of the finite part and some subsets of $\partial \mathbf{TP}^2$. Fix an affine patch and let f be the dehomogenization of F with respect to the patch. The *finite part* $V_{\mathbf{TP}^2}(F)^{\text{fin}}$ is the closure of $V_{\mathbf{R}^2}(f)$ in $\mathbf{TP}^2$.

2.2 Valuation and tropicalization

Definition 2.11. Let K be a field. A mapping val from K to $\mathbf{R} \cup \{\infty\}$ is a *valuation* if it satisfies the following axioms:

1. $a = 0 \iff \mathrm{val}(a) = \infty$,
2. $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$,
3. $\mathrm{val}(x + y) \geqslant \min\{\mathrm{val}(x), \mathrm{val}(y)\}$, with equality when $\mathrm{val}(x) \neq \mathrm{val}(y)$.

The proof of equality can be referred to [1]. If $\mathrm{val}(a) = 0, \forall a \in K$, val is said to be a *trivial* valuation on K. If we restrict the domain to $K^\times$, the image of val is an additive subgroup of $\mathbf{R}$ called the *value group* Γ_{val} of (K, val). An example is Puiseux series field $\mathbf{C}\{\{t\}\}$ with valuation taking the least exponent.

Definition 2.12. Let (K, val) be a valuation field. *Tropicalization of points* in K^n is defined by a mapping:

$$\mathrm{trop} : K^n \to \mathbf{T}^n : (x_i) \mapsto (-\,\mathrm{val}(x_i)), i = 1, \dots, n.$$

Let $f = \sum_{\mathbf{u} \in U \subset \mathbf{N}^n} c_{\mathbf{u}} x^{\mathbf{u}}$ be a classical polynomial in $K[x_1, \dots, x_n]$, where $U := \{\mathbf{u} \in \mathbf{N}^n : c_{\mathbf{u}} \neq 0\}$. The *tropicalization of* f is the piecewise linear function $\mathrm{trop}(f) : \mathbf{R}^n \to \mathbf{R}$ defined as:

$$\mathrm{trop}(f) = \bigoplus_{\mathbf{u} \in U \subset \mathbf{N}^n} (-\,\mathrm{val}(c_{\mathbf{u}})) \odot x^{\mathbf{u}}$$

The following theorem proofed by Kapranov establishes a connection between classical hypersurfaces over K and tropical hypersurfaces in $\mathbf{R}^n$.

Theorem 2.13. ([3]) *Assume that K is algebraically closed with a nontrivial valuation. f is defined as definition 2.12. Then tropical hypersurface defined by $trop(f)$ and the closure in Euclidean space $\mathbf{R}^n$ of tropicalization of classical algebraic hypersurface $V(f)$ conicide:*

$$V_{\mathbf{R}^n}(trop(f)) = \overline{trop(V(f))}$$

Remark 2.14. In this study, we work over a possibly not algebraically closed field K with a nontrivial valuation val.

2.3 Related polyhedral geometry

Definition 2.15. An intersection of finite closed half-spaces is called a *polyhedron*. Thus a polyhedron P is:

$$P := \{x \in \mathbf{R}^n : Ax \leqslant b\}, \text{ for some} A \in \mathbf{R}^{m \times n} \text{ and } b \in \mathbf{R}^m.$$

Definition 2.16. A *face* of a polyhedron P induced via a linear form $\mu \in (\mathbf{R}^n)^\vee$ is defined as:

$$face_\mu(P) := \{x \in P : x \cdot \mu \leqslant y \cdot \mu, \forall y \in P\}.$$

A face F of P is called *lower face* if for an arbitrary $x \in F$ and all $\alpha > 0$, $x - (0, \dots, 0, \alpha) \notin P$.

Definition 2.17. A collection Σ of polyhedra is called *polyhedral complex* if it satisfies:

1. If $P \in \Sigma$, then all its faces are in Σ.
2. If $P, Q \in \Sigma$ and $P \cap Q \neq \varnothing$, then $P \cap Q$ is a face of P as well as Q.

Remark 2.18. The polyhedron σ in Σ is called a *cell* of the complex. The *support* of Σ is a collection of points from all cells, which is denoted as $|\Sigma|$.

Definition 2.19. A *subdivision* of a polytope $P \subset \mathbf{R}^n$ is a polyhedral complex $\mathrm{SD}(P)$ if all cells are bounded and $|\mathrm{SD}(P)| = P$. $\mathrm{SD}(P)$ is *regular* if and only if it is induced by a polytope $P' \subset \mathbf{R}^{n+1}$ and satisfies:

1. $\pi(P') = P$, where $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$, $(x_1 \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$.
2. $\mathrm{SD}(P) = \{\pi(F) : F \text{ is a lower face of } P'\}$.

Definition 2.20. Let $f =$"$\sum_{\mathbf{u} \in U \subset \mathbf{N}^n} c_{\mathbf{u}} x^{\mathbf{u}}$"$\in \Gamma_{\mathrm{val}}[x_1, \ldots, x_n]$, where $U := \{\mathbf{u} \in \mathbf{N}^n : c_{\mathbf{u}} \neq -\infty\}$. *Newton polytope* of f is defined as the convex hull of U:

$$\mathrm{Newt}(f) := \mathrm{conv}(U).$$

2.4 The Structure Theorem

Theorem 2.21. ([1]) *Let* $f =$ " $\sum_{\mathbf{u} \in U \subset \mathbf{N}^n} c_{\mathbf{u}} x^{\mathbf{u}}$ " *be a tropical polynomial in* n *variables with coefficients in* Γ_{val}*, where* $U := \{\mathbf{u} \in \mathbf{N}^n : c_{\mathbf{u}} \neq -\infty\}$. $V_{\mathbf{R}^2}(f)$ *is connected and is the support of a polyhedral complex in* $\mathbf{R}^n$*, which is dual to regular subdivision of* $\mathrm{Newt}(f)$. *This regular subdivision is induced by* U *on the lattice points of* $\mathrm{Newt}(f)$.

Definition 2.22. For a given tropical $f \in \Gamma_{\mathrm{val}}[x, y]$. An *edge* of $V_{\mathbf{R}^2}(f)$ is dual to a 1-dimensional cell in the regular subdivision described above. A *node* of $V_{\mathbf{R}^2}(f)$ is dual to a 2-dimensional cell in the regular subdivision described above, which is the point in $V_{\mathbf{R}^2}(f)$ whose valence is more than two, i.e., it lies in more than two edges. The collection of all nodes of $V_{\mathbf{R}^2}(f)$ is denoted as $\mathrm{Node}(f)$.

Definition 2.23. For given tropical polynomials $f, g \in \Gamma_{\mathrm{val}}[x, y]$, let Σ_f and Σ_g be polyhedral complexes satisfying $|\Sigma_f| = V_{\mathbf{R}^2}(f)$ and $|\Sigma_g| = V_{\mathbf{R}^2}(g)$, respectively. $V_{\mathbf{R}^2}(f)$ and $V_{\mathbf{R}^2}(g)$ *intersect transversely* at $\mathbf{w}$ if there exist unique $\sigma_f \in \Sigma_f$ and $\sigma_g \in \Sigma_g$ such that $\mathbf{w} \in \mathrm{relint}(\sigma_f) \cap \mathrm{relint}(\sigma_g)$ and $\dim(\sigma_f + \sigma_g) = 2$.

Definition 2.24. ([1]) The *stable intersection* of $V_{\mathbf{R}^2}(f)$ and $V_{\mathbf{R}^2}(g)$ is defined as

$$V_{\mathbf{R}^2}(f) \cap_{st} V_{\mathbf{R}^2}(g) := \lim_{\varepsilon \to 0} V_{\mathbf{R}^2}(f) \cap (\varepsilon \mathbf{v} + V_{\mathbf{R}^2}(g)),$$

where $\mathbf{v}$ is an arbitrary vector in $\mathbf{R}^2$.

Remark 2.25. In $\mathbf{TP}^2$, we only consider the stable intersection of finite parts and give a similar definition.

3 Main Results

We demonstrate which points can belong to stable intersection of finite parts of two tropical projective plane curves in $\mathbf{TP}^2$.

Theorem 3.1. *Let* F, G *be tropical homogeneous polynomials in* $\Gamma_{\mathrm{val}}[x_1, x_2, x_3]$. *Fix an affine patch* U_3 *and let the dehomogenizations of* F, G *be* f, g *respectively and a map be* $\tau : \mathbf{R}^2 \to \mathbf{TP}^2, (x_1, x_2) \mapsto (x_1 : x_2 : 0)$. *Then we have*

$$\begin{aligned} V_{\mathbf{TP}^2}(F)^{\mathrm{fin}} \cap_{st} V_{\mathbf{TP}^2}(G)^{\mathrm{fin}} =& \tau(\mathrm{TI}(f, g)) \cup \tau(V_{\mathbf{R}^2}(f) \cap \mathrm{Node}(g)) \\ & \cup \tau(V_{\mathbf{R}^2}(g) \cap \mathrm{Node}(f)) \cup \mathrm{FP}_{\mathbf{TP}^2}(F, G). \end{aligned}$$

Here $\mathrm{TI}(f,g) := \{p \in \mathbf{R}^2 : V_{\mathbf{R}^2}(f)$ *and* $V_{\mathbf{R}^2}(g)$ *intersect transversally at* $p\}$. $\mathrm{Node}(f), \mathrm{Node}(g)$ *are node sets of* $V_{\mathbf{R}^2}(f), V_{\mathbf{R}^2}(g)$ *respectively.* $\mathrm{FP}_{\mathbf{TP}^2}(F,G) := \{(-\infty : -\infty : 0), (-\infty : 0 : -\infty), (0 : -\infty : -\infty)\} \cap V_{\mathbf{TP}^2}(F)^{\mathrm{fin}} \cap V_{\mathbf{TP}^2}(G)^{\mathrm{fin}}$.

We define that a point in the torus of $\mathbf{TP}^2$ is Γ_{val}-rational if its coordinates can be expressed as the product of an element from Γ_{val} and $\frac{1}{m}, m \in \mathbf{Z}_{>0}$. We show that the stable intersection points in theorem 3.1 with finite coordinates are Γ_{val}-rational.

Theorem 3.2. *If* $V_{\mathbf{TP}^2}(F)^{\mathrm{fin}} \cap_{st} V_{\mathbf{TP}^2}(G)^{\mathrm{fin}} \backslash \partial \mathbf{TP}^2 \neq \varnothing$, *then*

$$p \in V_{\mathbf{TP}^2}(F)^{\mathrm{fin}} \cap_{st} V_{\mathbf{TP}^2}(G)^{\mathrm{fin}} \backslash \partial \mathbf{TP}^2 \implies p \text{ is } \Gamma_{\mathrm{val}}\text{-rational.}$$

Moreover, $p = \frac{1}{i_p}(p_x : p_y : 0), p_x, p_y \in \Gamma_{\mathrm{val}}$ *where* i_p *is its intersection multiplicity.*

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